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Symmetric duality for nondifferentiable multiobjective fractional variational problems involving cones

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Abstract

We introduce a pair of symmetric dual problems for nondifferentiable multiobjective fractional variational problems with cone constraints over arbitrary cones. On the basis of weak efficiency, we obtain symmetric duality relations for Mond-Weir-type problems under invexity and pseudoinvexity assumptions. Our symmetric duality results extend and improve some known results in Mishra *et al.* (J. Math. Anal. Appl. 333:1093-1110, 2007) to the cone constraints.

MSC: 90C29; 90C32; 90C26

Keywords: symmetric duality; fractional variational problems; cone constraints; support functions

1 Introduction

The notion of symmetric duality in nonlinear programming, in which the dual of the dual is the primal, was first introduced by Dorn [1]. Dantzig *et al.* [2] discussed symmetric dual programs and established a symmetric duality under the convexity-concavity assumption. Mond and Hanson [3] first formulated a pair of symmetric dual variational problems by providing continuous analogue of the symmetric dual pair of Dantzig *et al.* [2] and proved the usual duality theorems under the convexity-concavity assumption. Suneja *et al.* [4] formulated a pair of Wolfe-type multiobjective symmetric dual programs over arbitrary cones, in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the functions involved to be cone-convex. Later on, Khurana [5] formulated a pair of Mond-Weir-type multiobjective symmetric dual programs over arbitrary cones and derived the symmetric duality theorems involving cone-pseudoinvex and strongly cone-pseudoinvex functions. Recently, Kim and Kim [6] extended the results of Suneja *et al.* [4] and Khurana [5] to nondifferentiable multiobjective symmetric dual programs for weak efficiency involving cone-invex and cone-pseudoinvex functions. Very recently, Ahmad *et al.* [7] extended the results of Suneja *et al.* [4] and Khurana [5] to a pair of multiobjective mixed symmetric dual programs over arbitrary cones. On the other hand, Chandra *et al.* [8] first introduced a symmetric duality in nonlinear fractional programming. Mond and Schechter [9] studied nondifferentiable symmetric duality, in which the objective function contains a support function. Following Mond and Schechter [9], Yang *et al.* [10] presented a pair of symmetric dual nonlinear fractional programming problems and established duality theorems under pseudo-convexity/pseudo-concavity assumptions on the kernel function. Further, Gulati *et al.* [11] generalized these results to static and continuous nonlinear fractional programming. For the multiobjective case of static nonlinear fractional program, symmetric duality was established under convexity assumptions. Subsequently, Gulati *et al.* [12] and Kim and Lee [13] gave two pairs of multiobjective symmetric dual variational programs, in which duality results were obtained under pseudoconvexity-pseudoconcavity and invexity assumptions, respectively. Chen [14] and Kim *et al.* [15] discussed duality results for multiobjective symmetric fractional variational programs involving invex functions. Recently, Mishra *et al.* [16] gave a symmetric dual pair for a class of nondifferentiable multiobjective fractional variational problems. Weak, strong, converse and self-duality relations were established under certain invexity assumptions. Recently, Ahmad *et al.* [17] formulated a pair of multiobjective fractional variational symmetric dual problems over cones and established duality theorems. Weak, strong and converse duality theorems are established under the generalized F-convexity assumptions. In this paper, we introduce a pair of symmetric duals for nondifferentiable multiobjective fractional variational problems with cone constraints over arbitrary cones. On the basis of weak efficiency, we obtain symmetric duality relations for Mond-Weir-type problems under invexity and pseudo-invexity assumptions. Our duality results extend the results in Mishra *et al.* [16] to the cone constraints over arbitrary cones with weak efficiency.

2 Preliminaries and notations

The following convention for vectors x and y in $\mathbb{R}^n$ will be used:

$$
\begin{aligned}
&x > y \quad \iff \quad x_i > y_i \quad \text{for all } i = 1, \ldots, n, \\
&x \geqq y \quad \iff \quad x_i \geqq y_i \quad \text{for all } i = 1, \ldots, n, \\
&x \geq y \quad \iff \quad x_i \geqq y_i \quad \text{for all } i = 1, \ldots, n, \quad \text{but} \quad x \neq y, \\
&x \not> y \quad \text{is the negation of } x > y.
\end{aligned}
$$

Throughout this paper, we will use the following notations.

Let $I = [a, b]$ be a real interval, let $f := (f_1, \ldots, f_k) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, $g := (g_1, \ldots, g_k) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x : I \to \mathbb{R}^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f by

$$
f_{ix} = \left[\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n} \right], \qquad f_{i\dot{x}} = \left[\frac{\partial f_i}{\partial \dot{x}_1}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n} \right], \quad i = 1, \ldots, k.
$$

Let $C(I, \mathbb{R}^m)$ denote the space of continuous functions $\phi : I \to \mathbb{R}^m$, with the uniform norm; $C_+(I, \mathbb{R}^m)$ is the cone of nonnegative functions in $C(I, \mathbb{R}^m)$. Denote by X the space of piecewise smooth functions $x : I \to \mathbb{R}^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$
u = Dx \quad \iff \quad x(t) = \alpha + \int_a^t u(s)\, ds,
$$

where α is a given boundary value: thus $D = d/dt$ except at discontinuities. For each $t \in I$, let $B_i(t)$ be a positive semidefinite $n \times n$ matrix with $B_i(\cdot)$ continuous on I, $i = 1, 2, \ldots, p$ and the symbol T denotes the transposition.

Consider the following multiobjective fractional variational problem:

$$
\begin{aligned}
&\text{(FVP)}\\
&\text{Minimize} \quad \frac{\int_a^b f(t,x(t),\dot{x}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t))\,dt}\\
&\qquad\qquad = \left(\frac{\int_a^b f_1(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_1(t,x(t),\dot{x}(t))\,dt},\dots,\frac{\int_a^b f_k(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_k(t,x(t),\dot{x}(t))\,dt}\right)\\
&\text{subject to} \quad x(a)=\alpha, \qquad x(b)=\beta,\\
&\qquad\qquad h\bigl(t,x(t),\dot{x}(t)\bigr) \leqq 0, \quad t \in I,
\end{aligned}
$$

where $h : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$.

Assume that $g_i(t,x(t),\dot{x}(t)) > 0$ and $f_i(t,x(t),\dot{x}(t)) \geqq 0$ for all $i = 1,2,\dots,k$. Let X denote the set of all feasible solutions of (FVP).

Definition 2.1 (1) A point $x^* \in X$ is said to be an efficient (Pareto optimal) solution of (FVP) if there exists no other feasible point $x \in X$ such that

$$\frac{\int_a^b f(t,x(t),\dot{x}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t))\,dt} \leq \frac{\int_a^b f(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g(t,x^*(t),\dot{x}^*(t))\,dt}.$$

(2) A point $x^* \in X$ is said to be a properly efficient solution of (FVP) if it is efficient for (FVP) and if there exists a scalar $M > 0$ such that, for all $i \in \{1,2,\dots,k\}$,

$$
\begin{aligned}
&\frac{\int_a^b f_i(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g_i(t,x^*(t),\dot{x}^*(t))\,dt} - \frac{\int_a^b f_i(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_i(t,x(t),\dot{x}(t))\,dt}\\
&\quad \leqq M\left(\frac{\int_a^b f_j(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_j(t,x(t),\dot{x}(t))\,dt} - \frac{\int_a^b f_j(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g_j(t,x^*(t),\dot{x}^*(t))\,dt}\right)
\end{aligned}
$$

for some $j \neq i$ such that

$$\frac{\int_a^b f_j(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_j(t,x(t),\dot{x}(t))\,dt} > \frac{\int_a^b f_j(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g_j(t,x^*(t),\dot{x}^*(t))\,dt}$$

whenever $x \in X$ and

$$\frac{\int_a^b f_i(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_i(t,x(t),\dot{x}(t))\,dt} < \frac{\int_a^b f_i(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g_i(t,x^*(t),\dot{x}^*(t))\,dt}.$$

(3) A point $x^* \in X$ is said to be a weakly efficient solution of (FVP) if there exists no other feasible point $x \in X$ such that

$$\frac{\int_a^b f(t,x(t),\dot{x}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t))\,dt} < \frac{\int_a^b f(t,x^*(t),\dot{x}^*(t))\,dt}{\int_a^b g(t,x^*(t),\dot{x}^*(t))\,dt}.$$

Now we recall the invexity for continuous case as follows.

Definition 2.2 The vector of functionals $\int_a^b f = (\int_a^b f_1, \dots, \int_a^b f_k)$ is said to be invex in x and $\dot{x}$ if for each $y : [a,b] \to \mathbb{R}^m$, with $\dot{y}$ piecewise smooth, there exists a function $\eta : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall i = 1, 2, \dots, k$

$$\begin{aligned}&\int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) - f_i(t,u,\dot{u},y,\dot{y})\}\,dt \\ &\quad \geqq \int_a^b \eta(t,x,\dot{x},u,\dot{u})^T \left[f_{ix}(t,u,\dot{u},y,\dot{y}) - \frac{d}{dt} f_{i\dot{x}}(t,u,\dot{u},y,\dot{y})\right] dt\end{aligned}$$

for all $x : [a,b] \to \mathbb{R}^n$, $u : [a,b] \to \mathbb{R}^n$, where $(\dot{x}(t), \dot{u}(t))$ is piecewise smooth on $[a,b]$.

Definition 2.3 The vector of functionals $-\int_a^b f = (-\int_a^b f_1, \dots, -\int_a^b f_k)$ is said to be invex in y and $\dot{y}$ if for each $x : [a,b] \to \mathbb{R}^n$, with $\dot{x}$ piecewise smooth, there exists function $\xi : [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \to \mathbb{R}^m$ such that $\forall i = 1, 2, \dots, k$,

$$\begin{aligned}&-\int_a^b \{f_i(t,x,\dot{x},v,\dot{v}) - f_i(t,x,\dot{x},y,\dot{y})\}\,dt \\ &\quad \geqq -\int_a^b \xi(t,v,\dot{v},y,\dot{y})^T \left[f_{iy}(t,x,\dot{x},y,\dot{y}) - \frac{d}{dt} f_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\right] dt\end{aligned}$$

for all $v : [a,b] \to \mathbb{R}^m$, $y : [a,b] \to \mathbb{R}^m$, where $(\dot{v}(t), \dot{y}(t))$ is piecewise smooth on $[a,b]$.

Definition 2.4 The vector of functionals $\int_a^b f = (\int_a^b f_1, \dots, \int_a^b f_k)$ is said to be pseudo-invex in x and $\dot{x}$ if for each $y : [a,b] \to \mathbb{R}^m$, with $\dot{y}$ piecewise smooth, there exists a function $\eta : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall i = 1, 2, \dots, k$,

$$\begin{aligned}&\int_a^b \eta(t,x,\dot{x},u,\dot{u})^T \left[f_{ix}(t,u,\dot{u},y,\dot{y}) - \frac{d}{dt} f_{i\dot{x}}(t,u,\dot{u},y,\dot{y})\right] dt \geqq 0 \\ &\quad \Rightarrow \quad \int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) - f_i(t,u,\dot{u},y,\dot{y})\}\,dt \geqq 0\end{aligned}$$

for all $x : [a,b] \to \mathbb{R}^n$, $u : [a,b] \to \mathbb{R}^n$, where $(\dot{x}(t), \dot{u}(t))$ is piecewise smooth on $[a,b]$.

Definition 2.5 The vector of functionals $-\int_a^b f$ is said to be pseudo-invex in y and $\dot{y}$ if for each $x : [a,b] \to \mathbb{R}^n$, with $\dot{x}$ piecewise smooth, there exists a function $\xi : [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $\forall i = 1, 2, \dots, k$,

$$\begin{aligned}&-\int_a^b \xi(t,v,\dot{v},y,\dot{y})^T \left[f_{iy}(t,x,\dot{x},y,\dot{y}) - \frac{d}{dt} f_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\right] dt \geqq 0 \\ &\quad \Rightarrow \quad -\int_a^b \{f_i(t,x,\dot{x},v,\dot{v}) - f_i(t,x,\dot{x},y,\dot{y})\}\,dt \geqq 0\end{aligned}$$

for all $v : [a,b] \to \mathbb{R}^m$, $y : [a,b] \to \mathbb{R}^m$, where $(\dot{v}(t), \dot{y}(t))$ is piecewise smooth on $[a,b]$.

We consider the problem of finding functions $x : [a,b] \to \mathbb{R}^n$ and $y : [a,b] \to \mathbb{R}^m$, where $(\dot{x}(t), \dot{y}(t))$ is piecewise smooth on $[a,b]$, to solve the following pair symmetric dual problems for nondifferentiable multiobjective fractional variational problems as follows.

(NFVP)

$$\begin{aligned}
\text{Minimize}\quad & \frac{\int_a^b \{f(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + s(x(t)|C) - y(t)^T z(t)\}\,dt}{\int_a^b \{g(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - s(x(t)|E) + y(t)^T r(t)\}\,dt} \\
&= \left(\frac{\int_a^b \{f_1(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + s(x(t)|C_1) - y(t)^T z_1(t)\}\,dt}{\int_a^b \{g_1(t,x(t),\dot{x}(t),y(t)) - s(x(t)|E_1) + y(t)^T r_1(t)\}\,dt}, \ldots, \right. \\
&\qquad \left. \frac{\int_a^b \{f_k(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + s(x(t)|C_k) - y(t)^T z_k(t)\}\,dt}{\int_a^b \{g_k(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - s(x(t)|E_k) + y(t)^T r_k(t)\}\,dt} \right) \\
\text{subject to}\quad & x(a) = 0 = x(b), \qquad y(a) = 0 = y(b), \\
& \dot{x}(a) = 0 = \dot{x}(b), \qquad \dot{y}(a) = 0 = \dot{y}(b), \\
& -\sum_{i=1}^k \tau_i \bigl\{ [f_{iy} - Df_{i\dot{y}} - z_i] G_i(x,y) - [g_{iy} - Dg_{i\dot{y}} + r_i] F_i(x,y) \bigr\} \in C_2^*, \\
& \int_a^b y(t)^T \sum_{i=1}^k \tau_i \bigl\{ [f_{iy} - Df_{i\dot{y}} - z_i] G_i(x,y) - [g_{iy} - Dg_{i\dot{y}} + r_i] F_i(x,y) \bigr\}\,dt \\
&\quad \geqq 0, \\
& \tau > 0, \quad \tau^T e = 1, \qquad x(t) \in C_1, \quad t \in I, \\
& z_i(t) \in D_i, \qquad r_i(t) \in H_i, \quad i = 1,2,\ldots,k.
\end{aligned}$$

(NFVD)

$$\begin{aligned}
\text{Maximize}\quad & \frac{\int_a^b \{f(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s(v(t)|D) + u(t)^T w(t)\}\,dt}{\int_a^b \{g(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + s(v(t)|H) - u(t)^T s(t)\}\,dt} \\
&= \left(\frac{\int_a^b \{f_1(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s(v(t)|D_1) + u(t)^T w_1(t)\}\,dt}{\int_a^b \{g_1(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + s(v(t)|H_1) - u(t)^T s_1(t)\}\,dt}, \ldots, \right. \\
&\qquad \left. \frac{\int_a^b \{f_k(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s(v(t)|D_k) + u(t)^T w_k(t)\}\,dt}{\int_a^b \{g_k(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + s(v(t)|H_k) - u(t)^T s_k(t)\}\,dt} \right) \\
\text{subject to}\quad & u(a) = 0 = u(b), \qquad v(a) = 0 = v(b), \\
& \dot{u}(a) = 0 = \dot{u}(b), \qquad \dot{v}(a) = 0 = \dot{v}(b), \\
& \sum_{i=1}^k \tau_i \bigl\{ [f_{iu} - Df_{i\dot{u}} + w_i] G_i^*(u,v) - [g_{iu} - Dg_{i\dot{u}} - s_i] F_i^*(u,v) \bigr\} \in C_1^*, \\
& \int_a^b u(t)^T \sum_{i=1}^k \tau_i \bigl\{ [f_{iu} - Df_{i\dot{u}} + w_i] G_i^*(u,v) - [g_{iu} - Dg_{i\dot{u}} - s_i] F_i^*(u,v) \bigr\}\,dt \\
&\quad \leqq 0, \\
& \tau > 0, \quad \tau^T e = 1, \qquad v(t) \in C_2, \quad t \in I, \\
& w_i(t) \in C_i, \qquad s_i(t) \in E_i, \quad i = 1,2,\ldots,k,
\end{aligned}$$

where $f_i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$ and $g_i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+ \setminus \{0\}$ are continuously differentiable functions, C_i, E_i $(1 \leqq i \leqq k)$ are a compact convex set in $\mathbb{R}^n$ and D_i, E_i $(1 \leqq i \leqq k)$ are a compact convex set in $\mathbb{R}^m$, C_1 and C_2 are closed convex cones in $\mathbb{R}^n$, $\mathbb{R}^m$ with nonempty interiors, respectively. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively, and $s(x|C_i) = \max\{\langle x, y\rangle | y \in C_i\}$. Let $h_i(x) = s(x|C_i)$, $i = 1, \ldots, p$. Then h_i is a convex function and $\partial h_i(x) = \{w \in C_i | \langle w, x\rangle = s(x|C_i)\}$, where ∂h_i is the subdifferential of h_i. Let

$$\begin{aligned}
F_i(x,y) &= \int_a^b \left\{ f_i(t,x,\dot{x},y,\dot{y}) + s\bigl(x(t)|C_i\bigr) - y(t)^T z_i \right\} dt; \\
G_i(x,y) &= \int_a^b \left\{ g_i(t,x,\dot{x},y,\dot{y}) - s\bigl(x(t)|E_i\bigr) + y(t)^T r_i \right\} dt; \\
F_i^*(u,v) &= \int_a^b \left\{ f_i(t,u,\dot{u},v,\dot{v}) - s\bigl(v(t)|D_i\bigr) + u(t)^T w_i \right\} dt;
\end{aligned}$$

and

$$G_i^*(u,v) = \int_a^b \left\{ g_i(t,u,\dot{u},v,\dot{v}) + s\bigl(v(t)|H_i\bigr) - u(t)^T s_i \right\} dt.$$

Let $f_x = f_x(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, $f_{\dot{x}} = (t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, *etc.* All the statements above for F_i, G_i, F_i^* and G_i^* will be assumed to hold for subsequent results. It is to be noted that

$$Df_{i\dot{y}} = f_{i\dot{y}y}\dot{y} + f_{i\dot{y}\dot{y}}\ddot{y} + f_{i\dot{y}x}\dot{x} + f_{i\dot{y}\dot{x}}\ddot{x}$$

and, consequently,

$$\begin{aligned}
&\frac{\partial}{\partial y} Df_{i\dot{y}} = Df_{i\dot{y}y}, \qquad \frac{\partial}{\partial \dot{y}} Df_{i\dot{y}} = Df_{i\dot{y}\dot{y}} + f_{i\dot{y}y}, \qquad \frac{\partial}{\partial \ddot{y}} Df_{i\dot{y}} = Df_{i\dot{y}\dot{y}}, \\
&\frac{\partial}{\partial x} Df_{i\dot{y}} = Df_{i\dot{y}x}, \qquad \frac{\partial}{\partial \dot{x}} Df_{i\dot{y}} = Df_{i\dot{y}\dot{x}} + f_{i\dot{y}x}, \qquad \frac{\partial}{\partial \ddot{x}} Df_{i\dot{y}} = Df_{i\dot{y}\dot{x}}.
\end{aligned}$$

In order to simplify the notations we introduce

$$p_i = \frac{F_i(x,y)}{G_i(x,y)} = \frac{\int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) + s(x(t)|C_i) - y(t)^T z_i\}\, dt}{\int_a^b \{g_i(t,x,\dot{x},y,\dot{y}) - s(x(t)|E_i) + y(t)^T r_i\}\, dt}, \quad i = 1, \ldots, k$$

and

$$q_i = \frac{F_i^*(u,v)}{G_i^*(u,v)} = \frac{\int_a^b \{f_i(t,u,\dot{u},v,\dot{v}) - s(v(t)|D_i) + u(t)^T w_i\}\, dt}{\int_a^b \{g_i(t,u,\dot{u},v,\dot{v}) + s(v(t)|H_i) - u(t)^T s_i\}\, dt}, \quad i = 1, \ldots, k$$

and express problems (NFVP) and (NFVD) equivalently as follows.

$$\begin{aligned}
\text{(NFVP)}' \quad &\text{Minimize} \quad p = (p_1, \ldots, p_k) \\
&\text{subject to} \quad x(a) = 0 = x(b), \qquad y(a) = 0 = y(b), \qquad (1) \\
&\qquad\qquad\quad \dot{x}(a) = 0 = \dot{x}(b), \qquad \dot{y}(a) = 0 = \dot{y}(b), \qquad (2)
\end{aligned}$$

$$\int_a^b \left\{f_i(t,x,\dot{x},y,\dot{y}) + s(x|C_i) - y^T z_i\right\} dt$$
$$- p_i \int_a^b \left\{g_i(t,x,\dot{x},y,\dot{y}) - s(x|E_i) + y^T r_i\right\} dt = 0,$$
$$i = 1,\dots,k, \tag{3}$$
$$-\sum_{i=1}^k \tau_i\left\{(f_{iy} - Df_{i\dot{y}} - z_i) - p_i(g_{iy} - Dg_{i\dot{y}} + r_i)\right\} \in C_2^*, \tag{4}$$
$$\int_a^b y(t)^T \sum_{i=1}^k \tau_i\left\{(f_{iy} - Df_{i\dot{y}} - z_i) - p_i(g_{iy} - Dg_{i\dot{y}} + r_i)\right\} dt$$
$$\geqq 0, \tag{5}$$
$$\tau > 0, \quad \tau^T e = 1, \qquad x(t) \in C_1, \quad t \in I, \tag{6}$$
$$z_i(t) \in D_i, \qquad r_i(t) \in H_i, \quad i = 1,2,\dots,k. \tag{7}$$

(NFVD)′ Maximize $q = (q_1,\dots,q_k)$

subject to
$$u(a) = 0 = u(b), \qquad v(a) = 0 = v(b), \tag{8}$$
$$\dot{u}(a) = 0 = \dot{u}(b), \qquad \dot{v}(a) = 0 = \dot{v}(b), \tag{9}$$
$$\int_a^b \left\{f_i(t,u,\dot{u},v,\dot{v}) - s(v|D_i) + u^T w_i\right\} dt$$
$$- q_i \int_a^b \left\{g_i(t,u,\dot{u},v,\dot{v}) + s(v|H_i) - u^T s_i\right\} dt = 0,$$
$$i = 1,\dots,k, \tag{10}$$
$$\sum_{i=1}^k \tau_i\left\{(f_{iu} - Df_{i\dot{u}} + w_i) - q_i(g_{iu} - Dg_{i\dot{u}} - s_i)\right\} \in C_1^*, \tag{11}$$
$$\int_a^b u(t)^T \sum_{i=1}^k \tau_i\left\{(f_{iu} - Df_{i\dot{u}} + w_i) - q_i(g_{iu} - Dg_{i\dot{u}} - s_i)\right\} dt$$
$$\leqq 0, \tag{12}$$
$$\tau > 0, \quad \tau^T e = 1, \qquad v(t) \in C_2, \quad t \in I, \tag{13}$$
$$w_i(t) \in C_i, \qquad s_i(t) \in E_i, \quad i = 1,2,\dots,k. \tag{14}$$

In the problems (NFVP)′ and (NFVD)′ above, it is to be noted that p and q are also nonnegative.

3 Duality theorems

In this section, we state duality theorems for problems (NFVP)′ and (NFVD)′, which lead to corresponding relations between (NFVP) and (NFVD). We establish weak, strong and converse duality relations between (NFVP)′ and (NFVD)′.

Theorem 3.1 (Weak duality) *Let* $(x(t), y(t), p, \tau, z(t), r(t))$ *be feasible for* (*NFVP*)′, *and let* $(u(t), v(t), q, \tau, w(t), s(t))$ *be feasible for* (*NFVD*)′. *Assume that* $\sum_{i=1}^k \tau_i \int_a^b \{(f_i + (\cdot)^T w_i) - q_i(g_i -$

$(\cdot)^T s_i)\}\,dt$ *is pseudo-invex in* x *and* $\dot{x}$ *with respect to* $\eta(x,u)$ *and* $-\sum_{i=1}^k \tau_i \int_a^b \{(f_i - (\cdot)^T z_i) - p_i(g_i + (\cdot)^T r_i)\}\,dt$ *is pseudo-invex in* y *and* $\dot{y}$ *with respect to* $\xi(v,y)$, *with* $\eta(x,u) + u \in C_1$ *and* $\xi(v,y) + y \in C_2$ $\forall t \in I$, *except possibly at corners of* $(\dot{x},\dot{y})$ *or* $(\dot{u},\dot{v})$. *Then* $p \nleq q$.

Proof From (11) and $\eta(x,u) + u \in C_1$, we get

$$\bigl(\eta(x,u)+u\bigr)^T \sum_{i=1}^k \tau_i \bigl\{(f_{iu} - Df_{i\dot{u}} + w_i) - q_i(g_{iu} - Dg_{i\dot{u}} - s_i)\bigr\}\,dt \geqq 0.$$

From (12),

$$\int_a^b \eta(x,u) \sum_{i=1}^k \tau_i \bigl\{(f_{iu} - Df_{i\dot{u}} + w_i) - q_i(g_{iu} - Dg_{i\dot{u}} - s_i)\bigr\}\,dt \geqq 0.$$

Since $\sum_{i=1}^k \tau_i \int_a^b \{(f_i + (\cdot)^T w_i) - q_i(g_i - (\cdot)^T s_i)\}\,dt$ is pseudo-invex with respect to $\eta(x,u)$, it follows that

$$\begin{aligned}&\sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,x,\dot{x},v,\dot{v}) + x^T w_i\bigr\} - q_i\bigl\{g_i(t,x,\dot{x},v,\dot{v}) - x^T s_i\bigr\}\bigr]\,dt \\ &\quad \geqq \sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,u,\dot{u},v,\dot{v}) + u^T w_i\bigr\} - q_i\bigl\{g_i(t,u,\dot{u},v,\dot{v}) - u^T s_i\bigr\}\bigr]\,dt. \end{aligned} \tag{15}$$

Since $x^T s_i \leq s(x|E_i)$, $s_i \in E_i$, and $x^T w_i \leq s(x|C_i)$, $w_i \in C_i$, (15) can be written as

$$\begin{aligned}&\sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,x,\dot{x},v,\dot{v}) + s(x|C_i)\bigr\} - q_i\bigl\{g_i(t,x,\dot{x},v,\dot{v}) - s(x|E_i)\bigr\}\bigr]\,dt \\ &\quad \geqq \sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,u,\dot{u},v,\dot{v}) + u^T w_i\bigr\} - q_i\bigl\{g_i(t,u,\dot{u},v,\dot{v}) - u^T s_i\bigr\}\bigr]\,dt. \end{aligned} \tag{16}$$

From (4) and $\xi(v,y) + y \in C_2$, we get

$$-\bigl(\xi(x,u)+y\bigr)^T \sum_{i=1}^k \tau_i \bigl\{(f_{iy} - Df_{i\dot{y}} - z_i) - p_i(g_{iy} - Dg_{i\dot{y}} + r_i)\bigr\}\,dt \geqq 0.$$

From (5),

$$-\int_a^b \xi(x,u)^T \sum_{i=1}^k \tau_i \bigl\{(f_{iy} - Df_{i\dot{y}} - z_i) - p_i(g_{iy} - Dg_{i\dot{y}} + r_i)\bigr\}\,dt \geqq 0.$$

By pseudo-invexity of $-\sum_{i=1}^k \tau_i \int_a^b \{(f_i - (\cdot)^T z_i) - p_i(g_i + (\cdot)^T r_i)\}\,dt$ with respect to $\xi(v,y)$, we get

$$\begin{aligned}&\sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,x,\dot{x},v,\dot{v}) - v^T z_i\bigr\} - p_i\bigl\{g_i(t,x,\dot{x},v,\dot{v}) + v^T r_i\bigr\}\bigr]\,dt \\ &\quad \leqq \sum_{i=1}^k \tau_i \int_a^b \bigl[\bigl\{f_i(t,x,\dot{x},y,\dot{y}) - y^T z_i\bigr\} - p_i\bigl\{g_i(t,x,\dot{x},y,\dot{y}) + y^T r_i\bigr\}\bigr]\,dt. \end{aligned}$$

Since $v^T r_i \le s(v|H_i)$, $r_i \in H_i$, and $v^T z_i \le s(v|D_i)$, $z_i \in D_i$,

$$\begin{aligned}
&-\sum_{i=1}^{k} \tau_i \int_a^b \big[\{f_i(t,x,\dot{x},v,\dot{v}) - s(v|D_i)\} - p_i\{g_i(t,x,\dot{x},v,\dot{v}) + s(v|H_i)\}\big]\,dt \\
&\quad \geqq -\sum_{i=1}^{k} \tau_i \int_a^b \big[\{f_i(t,x,\dot{x},y,\dot{y}) - y^T z_i\} - p_i\{g_i(t,x,\dot{x},y,\dot{y}) + y^T r_i\}\big]\,dt. \qquad (17)
\end{aligned}$$

From (16) and (17), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt \\
&\quad \geqq \sum_{i=1}^{k} \tau_i \bigg[\int_a^b \{f_i(t,u,\dot{u},v,\dot{v}) - s(v|D_i) + u^T w_i\}\,dt \\
&\quad - q_i \int_a^b \{g_i(t,u,\dot{u},v,\dot{v}) + s(v|H_i) - u^T s_i\}\,dt\bigg] \\
&\quad - \sum_{i=1}^{k} \tau_i \bigg[\int_a^b \{f_i(t,x,\dot{x},y,\dot{y}) + s(x|C_i) - y^T z_i\}\,dt \\
&\quad - p_i \int_a^b \{g_i(t,x,\dot{x},y,\dot{y}) - s(x|E_i) + y^T r_i\}\,dt\bigg]. \qquad (18)
\end{aligned}$$

From (3) and (10), (18) yields

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt \geqq 0. \qquad (19)$$

Suppose, if possible, that $p_i < q_i$ for all i, then from $\tau \geqq 0$, $\tau^T e = 1$ and $\int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt > 0$, $i = 1,2,\ldots,k$, we have

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t,x,\dot{x},v,\dot{v})\,dt < 0,$$

which contradicts (19), hence $p \nless q$. □

Consider the following multiobjective fractional variational problem.

$$\begin{aligned}
\text{(VP)}\quad \text{Minimize}\quad & \bigg(\int_a^b \big(f_1\big(t,x(t),\dot{x}(t)\big) + s\big(x(t)|D_1\big)\big)\,dt, \ldots, \\
& \int_a^b \big(f_k\big(t,x(t),\dot{x}(t)\big) + s\big(x(t)|D_k\big)\big)\,dt\bigg) \\
\text{subject to}\quad & x(a) = \alpha, \qquad x(b) = \beta, \\
& g_j\big(t,x(t),\dot{x}(t)\big) \leqq 0, \quad j = 1,\ldots,m, \\
& h_l\big(t,x(t),\dot{x}(t)\big) = 0, \quad l = 1,\ldots,p,
\end{aligned}$$

where $g_j : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function, $h_l : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function. Let $A = \{x \in X | x(a) = \alpha, x(b) = \beta, g_j(t, x(t), \dot{x}(t)) \leqq 0, j = 1, \dots, m, h_l(t, x(t), \dot{x}(t)) = 0, l = 1, \dots, p\}$.

We need the following Fritz John necessary optimality condition in order to establish a strong duality theorem. Using the proof of Theorem 1 in [18], we obtain the following theorem.

$$
\begin{aligned}
(\mathrm{VP})' \quad & \text{Minimize} \quad F(x) + J(x) := \big(F_1(x) + J_1(x), \dots, F_k(x) + J_k(x)\big) \\
& \text{subject to} \quad G(x) \in C_+\big(I, \mathbb{R}^m\big), \\
& \qquad\qquad\quad H(x) = 0,
\end{aligned}
$$

where $F_i : X \to \mathbb{R}$ are functions defined on $x \in X$, $F_i(x) = \int_a^b f_i(t, x(t), \dot{x}(t))\,dt$, $J_i : X \to \mathbb{R}$ are functions defined by $J_i(x) = \int_a^b s(x(t)|D_i)\,dt$, $G : X \to C(I, \mathbb{R}^m)$ are functions defined by $G(x)(t) = (g_1(t, x(t), \dot{x}(t)), \dots, g_m(t, x(t), \dot{x}(x)))$ and $H : X \to C(I, \mathbb{R}^p)$ are functions defined by $H(x)(t) = (h_1(t, x(t), \dot{x}(t)), \dots, h_p(t, x(t), \dot{x}(t)))$ and $C_+(I, \mathbb{R}^m)$. Let $S = \{x \in X | G(x) \in C_+(I, \mathbb{R}^m), H(x) = 0\}$.

Theorem 3.2 *Let $\bar{x} \in A$ be a weakly efficient solution of* (VP). *Suppose that there exists an $\hat{x} \in X$ such that $G(\bar{x}) + G'(\bar{x})(\hat{x} - \bar{x}) \in -\int C_+(I, \mathbb{R}^m)$, $H'(\bar{x})(\hat{x} - \bar{x}) = 0_{C(I,\mathbb{R}^p)}$, and the map $H'(\bar{x})$ is surjective. Then there exist $\tau_i \geqq 0$, $(\tau_1, \dots, \tau_k) \neq 0$ and piecewise smooth $\lambda : I \to \mathbb{R}^m_+$, $\mu : I \to \mathbb{R}^p$, and $w_i \in D_i$, $i = 1, \dots, k$, satisfying*

$$
\begin{aligned}
& \sum_{i=1}^k \tau_i\big[f_{ix}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) + w_i\big] + \sum_{j=1}^m \lambda_j(t) g_{jx}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) + \sum_{l=1}^p \mu_l(t) h_{lx}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) \\
& \quad = D\left[\sum_{i=1}^k \tau_i f_{i\dot{x}}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) + \sum_{j=1}^m \lambda_j(t) g_{j\dot{x}}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) + \sum_{l=1}^p \mu_l(t) h_{l\dot{x}}\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big)\right], \\
& \sum_{j=1}^m \lambda_j(t) g_j\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) = 0, \\
& w_i^T x(t) = s\big(x(t)|D_i\big), \quad i = 1, \dots, k
\end{aligned}
$$

for all $t \in I$.

Theorem 3.3 (Strong duality) *Let $(x_0(t), y_0(t), p_0, \tau_0, z_0(t), r_0(t))$ be a weakly efficient solution for* (NFVP)′ *and fix $\tau = \tau_0$ in* (NFVD)′, *and define $p_{0i} = \frac{\int_a^b \{f_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) + s(x_0|C_i) - y_0^T z_{0i}\}\,dt}{\int_a^b \{g_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) - s(x_0|E_i) + y_0^T r_{0i}\}\,dt}$, $i = 1, 2, \dots, k$. Suppose that all the conditions in weak duality are fulfilled. Furthermore, assume that*

$$
\text{(I)} \quad p_{0i} > 0, \quad i = 1, \dots, k,
$$

$$
\begin{aligned}
\text{(II)} \quad & \sum_{i=1}^k \tau_{0i} \int_a^b \Psi(t)^T \big[\big\{(f_{iyy} - p_{0i} g_{iyy}) - D(f_{iy\dot{y}} - p_{0i} g_{iy\dot{y}})\big\} \\
& \quad - D\big\{(f_{iy\dot{y}} - D f_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - p_{0i}(g_{iy\dot{y}} - D g_{i\dot{y}\dot{y}} - g_{i\dot{y}y})\big\} \\
& \quad + D^2\big\{-(f_{i\dot{y}\dot{y}} - p_{0i} g_{i\dot{y}\dot{y}})\big\}\big]\Psi(t)\,dt \leqq 0
\end{aligned}
$$

implies that $\Psi(t) = 0$, $\forall t \in I$, *and*

$$\text{(III)}\quad \left(\int_a^b \bigl\{(f_{1y} - Df_{1\dot{y}} - z_{01}) - p_{01}(g_{1y} - Dg_{1\dot{y}} + r_{01})\bigr\}\, dt, \ldots, \int_a^b \bigl\{(f_{ky} - Df_{k\dot{y}} - z_{0k}) - p_{0k}(g_{ky} - Dg_{k\dot{y}} + r_{0k})\bigr\}\, dt\right)$$

is linearly independent.

Then there exist $w_{0i}(t) \in C_i$, $s_{0i}(t) \in E_i$, $i = 1, 2, \ldots, k$ *such that* $(x_0(t), y_0(t), p_0, \tau_0, w_0(t), s_0(t))$ *is weakly efficient solution of* (NFVD)$'$.

Proof Since $(x_0(t), y_0(t), p_0, \tau_0, z_0(t), r_0(t))$ is a weakly efficient solution of (NFVP)$'$, by Theorem 3.2, there exist $\lambda \in \mathbb{R}^k$, $\alpha \in \mathbb{R}^k$, $\zeta \in \mathbb{R}$, $\mu : I \longrightarrow \mathbb{R}^k$, piecewise smooth $\beta(t) : I \longrightarrow C_2$ and $\rho(t) : I \longrightarrow C_1^*$ such that

$$\lambda_i - \alpha_i\bigl(g_i - s_{0i} + y_0^T r_{0i}\bigr) - (g_{iy} - Dg_{i\dot{y}} + r_{0i})(\beta - \zeta y_0) = 0, \quad i = 1, 2, \ldots, k, \tag{20}$$

$$\begin{aligned}&\sum_{i=1}^k \alpha_i\bigl[\bigl\{(f_{ix} + w_{1i}) - p_{0i}(g_{ix} - s_{0i})\bigr\} - D(f_{i\dot{x}} - p_{0i}g_{i\dot{x}})\bigr]\\ &\quad + \sum_{i=1}^k \tau_{0i}\bigl[\bigl\{(f_{iyx} - p_{0i}g_{iyx}) - D(f_{i\dot{y}x} - p_{0i}g_{i\dot{y}x})\bigr\}\\ &\quad - D\bigl\{(f_{iy\dot{x}} - Df_{i\dot{y}\dot{x}} - f_{i\dot{y}x}) - p_{0i}(g_{iy\dot{x}} - Dg_{i\dot{y}\dot{x}} - g_{i\dot{y}x})\bigr\}\\ &\quad + D^2\bigl\{-(f_{i\dot{y}\dot{x}} - p_{0i}g_{i\dot{y}\dot{x}})\bigr\}\bigr](\beta - \zeta y_0) - \rho = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\sum_{i=1}^k (\alpha_i - \zeta\tau_{0i})\bigl\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\bigr\}\\ &\quad + \sum_{i=1}^k \tau_{0i}\bigl[\bigl\{(f_{iyy} - p_{0i}g_{iyy}) - D(f_{i\dot{y}y} - p_{0i}g_{i\dot{y}y})\bigr\}\\ &\quad - D\bigl\{(f_{iy\dot{y}} - Df_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - p_{0i}(g_{iy\dot{y}} - Dg_{i\dot{y}\dot{y}} - g_{i\dot{y}y})\bigr\}\\ &\quad + D^2\bigl\{-(f_{i\dot{y}\dot{y}} - p_{0i}g_{i\dot{y}\dot{y}})\bigr\}\bigr](\beta - \zeta y_0) = 0,\end{aligned} \tag{22}$$

$$\bigl\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\bigr\}(\beta - \zeta y_0) - \mu_i = 0, \quad i = 1, \ldots, k, \tag{23}$$

$$\alpha_i p_{0i} y_0 + (\beta - \zeta y_0)\tau_{0i} p_{0i} \in N_{H_i}(r_{0i}), \quad i = 1, 2, \ldots, k, \tag{24}$$

$$\alpha_i y_0 + (\beta - \zeta y_0)\tau_{0i} \in N_{D_i}(z_{0i}), \quad i = 1, 2, \ldots, k, \tag{25}$$

$$\sum_{i=1}^k \alpha_i\left[\int_a^b \bigl\{\bigl(f_i + s(x_0|C_i) - y_0^T z_{0i}\bigr) - p_{0i}\bigl(g_i - s(x_0|E_i) + y_0^T r_{0i}\bigr)\bigr\}\, dt\right] = 0, \tag{26}$$

$$\beta^T \sum_{i=1}^k \tau_{0i}\bigl\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\bigr\} = 0, \tag{27}$$

$$\zeta y_0^T \sum_{i=1}^{k} \tau_{0i} \int_a^b \big\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\big\}\, dt = 0, \tag{28}$$

$$\mu^T \tau_0 = 0, \tag{29}$$

$$\rho^T x_0 = 0, \tag{30}$$

$$w_{1i} \in C_i, \qquad w_{1i}^T x_0 = s(x_0|C_i), \quad i = 1, 2, \ldots, k, \tag{31}$$

$$s_{0i} \in E_i, \qquad s_{0i}^T x_0 = s(x_0|E_i), \quad i = 1, 2, \ldots, k, \tag{32}$$

$$\big(\lambda, \alpha, \beta(t), \zeta, \mu, \rho(t)\big) \geqq 0, \tag{33}$$

$$\big(\lambda, \alpha, \beta(t), \zeta, \mu, \rho(t)\big) \neq 0. \tag{34}$$

Multiplying (22) by $(\beta - \zeta y_0)^T$,

$$\begin{aligned}
&\sum_{i=1}^{k} (\alpha_i - \zeta \tau_{0i})\big\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\big\}(\beta - \zeta y_0) \\
&\quad + (\beta - \zeta y_0)^T \sum_{i=1}^{k} \tau_{0i}\big[\big\{(f_{iyy} - p_{0i} g_{iyy}) - D(f_{i\dot{y}y} - p_{0i} g_{i\dot{y}y})\big\} \\
&\quad - D\big\{(f_{iy\dot{y}} - Df_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - p_{0i}(g_{iy\dot{y}} - Dg_{i\dot{y}\dot{y}} - g_{i\dot{y}y})\big\} \\
&\quad + D^2\big\{-(f_{i\dot{y}\dot{y}} - p_{0i} g_{i\dot{y}\dot{y}})\big\}\big](\beta - \zeta y_0) = 0.
\end{aligned}$$

Using the result in equality (23) and (29), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \alpha_i \mu_i + (\beta - \zeta y_0)^T \sum_{i=1}^{k} \tau_{0i}\big[\big\{(f_{iyy} - p_{0i} g_{iyy}) - D(f_{i\dot{y}y} - p_{0i} g_{i\dot{y}y})\big\} \\
&\quad - D\big\{(f_{iy\dot{y}} - Df_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - p_{0i}(g_{iy\dot{y}} - Dg_{i\dot{y}\dot{y}} - g_{i\dot{y}y})\big\} \\
&\quad + D^2\big\{-(f_{i\dot{y}\dot{y}} - p_{0i} g_{i\dot{y}\dot{y}})\big\}\big](\beta - \zeta y_0) = 0.
\end{aligned}$$

Since $\alpha \in \mathbb{R}_+^k$, $\mu \in \mathbb{R}_+^k$, $\alpha^T \mu \geqq 0$, and hence

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_{0i} \int_a^b (\beta - \zeta y_0)^T \big[\big\{(f_{iyy} - p_{0i} g_{iyy}) - D(f_{i\dot{y}y} - p_{0i} g_{i\dot{y}y})\big\} \\
&\quad - D\big\{(f_{iy\dot{y}} - Df_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - p_{0i}(g_{iy\dot{y}} - Dg_{i\dot{y}\dot{y}} - g_{i\dot{y}y})\big\} \\
&\quad + D^2\big\{-(f_{i\dot{y}\dot{y}} - p_{0i} g_{i\dot{y}\dot{y}})\big\}\big](\beta - \zeta y_0)\, dt \leqq 0.
\end{aligned}$$

Which by virtue of the hypothesis (II) yields

$$\beta = \zeta y_0 \quad \forall t \in I. \tag{35}$$

From (22) along with (35), we obtain

$$\sum_{i=1}^{k} (\alpha_i - \zeta \tau_{0i}) \int_a^b \big\{(f_{iy} - Df_{i\dot{y}} - z_{0i}) - p_{0i}(g_{iy} - Dg_{i\dot{y}} + r_{0i})\big\}\, dt = 0.$$

By hypothesis (III),

$$\alpha_i = \zeta \tau_{0i}, \quad i = 1, \dots, k. \tag{36}$$

If $\zeta = 0$, then (36) implies that $\alpha = 0$ and using (35) $\beta = 0$. From (20), we get $\lambda = 0$, and from (21) $\rho = 0$ and using (23), we get that $\mu = 0$, which contradicts (34). Hence $\zeta > 0$ and $\alpha > 0$. Hence by (35), $y_0 \in C_2$ $\forall t \in I$. By (21), (35) and $\alpha_i = \zeta \tau_{0i}$, $i = 1, \dots, k$,

$$\sum_{i=1}^{k} \zeta \tau_{0i} \big[\big\{ (f_{ix} + w_{1i}) - p_{0i}(g_{ix} - s_{0i}) \big\} - D(f_{i\dot{x}} - p_{0i} g_{i\dot{x}}) \big] = \rho \in C_1^*. \tag{37}$$

Since $\sum_{i=1}^{k} \tau_{0i}[\{(f_{ix} + w_{1i}) - p_{0i}(g_{ix} - s_{0i})\} - D(f_{i\dot{x}} - p_{0i} g_{i\dot{x}})] \in C_1^*$. By multiplying both sides of equation (37) by x_0, hence from (30) we get,

$$\int_a^b x_0^T \sum_{i=1}^{k} \tau_{0i} \big[\big\{ (f_{ix} + w_{1i}) - p_{0i}(g_{ix} - s_{0i}) \big\} - D(f_{i\dot{x}} - p_{0i} g_{i\dot{x}}) \big] \, dt = 0.$$

Equation (26) with $\alpha > 0$ implies that

$$\begin{aligned} \int_a^b & \big\{ \big(f_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) + s(x_0 | C_i) - y_0^T z_{0i} \big) \\ & - p_{0i} \big(g_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) - s(x_0 | E_i) + y_0^T r_{0i} \big) \big\} \, dt = 0. \end{aligned} \tag{38}$$

By (25) and the fact that $\beta = \zeta y_0$, $\alpha_i y_0 \in N_{D_i}(z_{0i})$, $i = 1, \dots, k$. Since $\alpha_i > 0$, and so $y_0 \in N_{D_i}(z_{0i})$, hence $y_0^T z_{0i} = s(y_0 | D_i)$, $i = 1, \dots, k$. By (24) and the fact that $\beta = \zeta y_0$, $\alpha_i p_{0i} y_0 \in N_{H_i}(r_{0i})$, $i = 1, \dots, k$. Since $\alpha_i > 0$, $p_{0i} > 0$, and so $y_0 \in N_{H_i}(r_{0i})$, hence $y_0^T r_{0i} = s(y_0 | H_i)$, $i = 1, \dots, k$. Thus, from (31), (32) and $y_0^T z_{0i} = s(y_0 | D_i)$, $y_0^T r_{0i} = s(y_0 | H_i)$, $i = 1, \dots, k$, equation (38) implies

$$\begin{aligned} \int_a^b & \big\{ \big(f_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) - s(y_0 | D_i) + x_0^T w_{1i} \big) \\ & - p_{0i} \big(g_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) + s(y_0 | H_i) - x_0^T s_{0i} \big) \big\} \, dt = 0, \end{aligned}$$

and

$$\begin{aligned} p_{0i} &= \frac{\int_a^b \{ f_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) + s(x_0 | C_i) - y_0^T z_{0i} \} \, dt}{\int_a^b \{ g_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) - s(x_0 | E_i) + y_0^T r_{0i} \} \, dt} \\ &= \frac{\int_a^b \{ f_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) - s(y_0 | D_i) + x_0^T w_{1i} \} \, dt}{\int_a^b \{ g_i(t, x_0, \dot{x}_0, y_0, \dot{y}_0) + s(y_0 | H_i) - x_0^T s_{0i} \} \, dt} \\ &= q_{0i}. \end{aligned}$$

Thus $(x_0, y_0, p_0, \tau_0, z_0, r_0)$ is feasible for (NFVD)$'$, and the objective values of (NFVP)$'$ and (NFVD)$'$ are equal there. Clearly, $(x_0, y_0, p_0, \tau_0, z_0, r_0)$ is weakly efficient for (NFVD)$'$. If $(x_0, y_0, p_0, \tau_0, z_0, r_0)$ is not weakly efficient for (NFVD)$'$, then for some feasible $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{\tau}, \tilde{z}, \tilde{r})$ of (NFVD)$'$, there exist $w_{0i}(t) \in C_i$, $s_{0i}(t) \in E_i$ such that $p_0 < \tilde{p}$, with $\widetilde{p}_i =$

$\frac{\int_a^b \{f_i(t,\tilde{x},\dot{\tilde{x}},\tilde{y},\dot{\tilde{y}})-s(\tilde{y}|D_i)+\tilde{x}^T w_{0i}\}\,dt}{\int_a^b \{g_i(t,\tilde{x},\dot{\tilde{x}},\tilde{y},\dot{\tilde{y}})+s(\tilde{y}|H_i)-\tilde{x}^T s_{0i}\}\,dt}$, $i = 1,\dots,k$. Since $g_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0) > 0$, $i = 1,\dots,k$, it follows that $\sum_{i=1}^k \tau_i(p_{0i} - \tilde{p}_i)\int_a^b g_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0)\,dt < 0$, which contradicts by weak duality, equation (19). Thus $(x_0,y_0,p_0,\tau_0,z_0,r_0)$ is a weakly efficient solution of (NFVD)$'$. Hence the result holds. □

Theorem 3.4 (Converse duality) *Let* $(x_0(t),y_0(t),q_0,\tau_0,w_0(t),s_0(t))$ *be a weakly efficient solution for* (NFVD)$'$ *and fix* $\tau = \tau_0$ *in* (NFVP)$'$, *and define*

$$q_{0i} = \frac{\int_a^b \{f_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0) - s(y_0|D_i) + x_0^T w_{0i}\}\,dt}{\int_a^b \{g_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0) + s(y_0|H_i) - x_0^T s_{0i}\}\,dt}, \quad i = 1,2,\dots,k.$$

Suppose that all the conditions in weak duality are fulfilled. Furthermore, assume that

$$\text{(I)}\quad q_{0i} > 0, \quad i = 1,\dots,k,$$

$$\begin{aligned}\text{(II)}\quad &\sum_{i=1}^k \tau_{0i}\int_a^b \Psi(t)^T\Big[\big\{(f_{ixx} - q_{0i}g_{ixx}) - D(f_{i\dot{x}x} - q_{0i}g_{i\dot{x}x})\big\}\\ &\quad - D\big\{(f_{ix\dot{x}} - Df_{i\dot{x}\dot{x}} - f_{i\dot{x}x}) - q_{0i}(g_{ix\dot{x}} - Dg_{i\dot{x}\dot{x}} - g_{i\dot{x}x})\big\}\\ &\quad + D^2\big\{-(f_{i\dot{x}\dot{x}} - q_{0i}g_{i\dot{x}\dot{x}})\big\}\Big]\Psi(t)\,dt \geqq 0\end{aligned}$$

implies that $\Psi(t) = 0$, $\forall t \in I$, *and*

$$\begin{aligned}\text{(III)}\quad &\bigg(\int_a^b \big\{(f_{1x} - Df_{1\dot{x}} + w_{01}) - q_{01}(g_{1x} - Dg_{1\dot{x}} - s_{01})\big\}\,dt,\dots,\\ &\quad\int_a^b \big\{(f_{kx} - Df_{k\dot{x}} + w_{0k}) - q_{0k}(g_{kx} - Dg_{k\dot{x}} - s_{0k})\big\}\,dt\bigg)\end{aligned}$$

is linearly independent.

Then there exist $z_{0i}(t) \in D_i$, $r_{0i}(t) \in H_i$, $i = 1,2,\dots,k$ *such that* $(x_0(t),y_0(t),q_0,\tau_0,z_0(t),r_0(t))$ *is weakly efficient solution of* (NFVP)$'$.

Proof It is analogous to the proof of the lines of Theorem 3.3. □

Remark 3.1 (1) When $C = D = E = H = \{0\}$, then the support functions and inner products in the problems (NFVP) and (NFVD) in the draft disappear, and hence (NFVP) and (NFVD) in the draft collapse to (P) and (D) in the paper of Ahmad, Sharma (EJOR, Vol. 188, 2008, pp. 695-704) [19].

(2) When $C_1 = \mathbb{R}_+^n$, and $C_2 = \mathbb{R}_+^m$, then the problems (NFVP) and (NFVD) reduce to those considered by Mishra *et al.* [16], respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DSK introduced a pair of symmetric dual problems for nondifferentiable multiobjective fractional variational problems with cone constraints and established symmetric duality relations for Mond-Weir-type problems under invexity and pseudoinvexity assumptions. YMK and MHK carried out the symmetric duality studies for nondifferentiable multiobjective fractional variational problems, participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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